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A Note on Invariant Observables

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The ergodic theory and particularly the individual ergodic theorem were studied in many structures. Recently the individual ergodic theorem has been proved for MV-algebras of fuzzy sets (Riečan, 2000; Riečan and Neubrunn, 1997) and even in general MV-algebras (Jurečková, 2000). The notion of almost everywhere equality of observables was introduced by B. Riečan and M. Jurečková in Riečan and Jurečková (2005). They proved that the limit of Cesaro means is an invariant observable for P-observables. In this paper show that the assumption of P-observable can be omitted.

KEY WORDS: the invariant observable; the Cesaro means; the individual ergodic theory; the almost everywhere equality.

1. INTRODUCTION

The ergodic theory and particularly the individual ergodic theorem were studied in many structures (Dvurečenskij and Riečan, 1980; Harman, 1985; Harman and Riečan, 1992; Jurečková, 2000; Lutterová and Pulmannová, 1985; Petersen, 1983; Pulmannová, 1982; Riečan, 1982; Riečan, 2000; Riečan and Mundici, 2002; Riečan and Neubrunn, 1997; Vrábel, 1988; Walters, 1975). Recently the individual ergodic theorem has been proved for MV-algebras of fuzzy sets (Riečan, 2000; Riečan and Neubrunn, 1997) and even in general MV-algebras (Jurečková, 2000).

In classical probability space $(\Omega, \mathcal{S}, P)$ the individual ergodic theorem (Petersen, 1983; Walters, 1975) guarantees the existence of a random variable $\xi^* : \Omega \to \mathbb{R}$ satisfying the following conditions:

(i) ξ^* is integrable and $E(\xi^*) = E(\xi)$,
(ii) $\frac{1}{n}\sum_{i=0}^{n-1} \xi \circ T \to \xi^*$ P-almost everywhere,
(iii) $\xi^* \circ T = \xi^*$ P-almost everywhere,

where $T : \Omega \to \Omega$ is a measure preserving map, i.e., $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for each $A \in \mathcal{S}$, $\xi : \Omega \to \mathbb{R}$ is an integrable random variable with the mean value $E(\xi)$.

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In connection with the generalization of property (iii) in the individual ergodic theorem, B. Riečan and M. Jurečková introduced the notion of almost everywhere equality of observables (Riečan and Jurečková, 2005). They assumed the P-observables and proved that the limit of Cesaro means is an invariant observable. In this paper, we will show that the assumption of P-observable can be omitted.

2. BASIC NOTIONS

In this section, we introduce the basic notions and theorems. They can be found in (Riečan and Neubrunn, 1997). We consider the fuzzy quantum logic

$$\mathcal{F} = \{f : \Omega \to \langle 0, 1\rangle;\ f \text{ is } \mathcal{S} - \text{measurable}\}.$$

The corresponding notion to the notion of a random variable is an observable. ***An observable*** is a mapping $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ such that:

(O1) $x(\mathbb{R}) = 1_{\mathcal{F}}$,
(O2) if $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$,
(O3) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Instead of a probability measure in the Kolmogorov model there is considered a state in $\mathcal{F}$. ***A state*** is a mapping $m : \mathcal{F} \to \langle 0, 1\rangle$ such that:

(S1) $m(1_{\mathcal{F}}) = 1$
(S2) if $f + g \le 1_{\mathcal{F}}$, then $m(f + g) = m(f) + m(g)$,
(S3) if $f_n \nearrow f$, then $m(f_n) \nearrow m(f)$.

The next notion of the joint observable corresponds to the notion of the random vector in classical probability theory. Let $x, y : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ be two observables.

The joint observable of the observables x, y is a mapping $h : \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ satisfying following conditions:

(JO1) $h(\mathbb{R}^2) = 1_{\mathcal{F}}$,
(JO2) if $A \cap B = \emptyset$, then $h(A \cup B) = h(A) + h(B)$,
(JO3) if $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$,
(JO4) $h(C \times D) = x(C) \cdot y(D)$, $C, D \in \mathcal{B}(\mathbb{R})$.

Recall that in $\mathcal{F}$ for each pair of observables x, y their joint observable exists (see Riečan and Neubrunn, 1997 *Theorem 8.3.2*).

The transformation $T : \Omega \to \Omega$ is also replaced by a mapping $\tau : \mathcal{F} \to \mathcal{F}$.

An m-preserving transformation is a mapping $\tau : \mathcal{F} \to \mathcal{F}$ satisfying the following conditions:

(T1) $\tau(1_{\mathcal{F}}) = 1_{\mathcal{F}}$

(T2) if $f + g \leq 1_{\mathcal{F}}$, then $\tau(f + g) = \tau(f) + \tau(g)$,
(T3) if $f_n \nearrow f$, then $\tau(f_n) \nearrow \tau(f)$
(T4) $\tau(f) \cdot \tau(g) = \tau(f \cdot g)$,
(T5) $\tau(f \wedge g) = \tau(f) \wedge \tau(g)$,
(T6) $m(\tau(f)) = m(f)$.

The next important notion is notion of almost everywhere coincidence introduced in Riečan and Jurečková (2005). Let m be a state on $\mathcal{F}$. We say that ***observables*** $y, z : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ ***coincide*** m***-almost everywhere***, i.e. $y = z$ m-almost everywhere, if

$$m(h(\Delta)) = 1$$

where $\Delta = \{(u, v) \in \mathbb{R}^2; u = v\}$ and $h : \mathcal{B}(\mathbb{R}^2) \to \mathcal{F}$ is the joint observable of y, z.

This notion doesn't depend on the choice of the joint observable h.

Theorem 2.1. Riečan and Jurečková (2005) *The observables y,z coincide m-almost everywhere if and only if*

$$m\left(y((-\infty, u)) \cdot z((u, \infty))\right) = 0 \text{ and } m\left(y((u, \infty)) \cdot z((-\infty, u))\right) = 0$$

for each $u \in \mathbb{R}$.

3. INVARIANT OBSERVABLES

In this section we prove that the limit of Cesaro means is an invariant observable for each integrable observable x. Our motivation is the individual ergodic theorem (see Riečan and Neubrunn, 1997 *Theorem 8.7.2*).

Individual Ergodic Theorem. Let x be an integrable observable and τ be an m-preserving transformation. Then there is an integrable observable x^* satisfying the following conditions:

(i) $E(x^*) = E(x)$;
(ii) $\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x \to x^* m$-almost everywhere.

First we start with the Kolmogorov construction.

Let h_n be the joint observable of observables $x, \tau \circ x, \ldots, \tau^n \circ x$. The system

$$\{P_n = m \circ h_n, n \in \mathbb{N}\}$$

is a consistent system of probability measures. By the Kolmogorov theorem there exists a probability measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ such that

$$P\left(\Pi_n^{-1}(A)\right) = P_n(A)$$

for each $A \in, \mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$, where $\Pi_n : \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to \mathcal{B}(\mathbb{R}^n)$ is the projection defined by $\Pi_n((u_i)_1^\infty) = (u_1, \ldots, u_n)$

Define the measure preserving transformation $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$T((u_i)_1^\infty) = (v_i)_1^\infty, \; v_i = u_{i+1}$$

and the first coordinate random variable $\xi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ by

$$\xi((u_i)_1^\infty) = u_1.$$

Let $g_n : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function defined by

$$g_n(u_1, \ldots, u_n) = \frac{1}{n} \sum_{i=1}^{n} u_i.$$

Theorem 3.1. Riečan and Neubrunn (1997) *Let* $y_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x = h_n \circ g_n^{-1}$, $\eta_n = \frac{1}{n} \sum_{i=0}^{n-1} \xi \circ T^i$. *Then the sequence of observables* $(y_n)_n$ *converges m-almost everywhere to an observable y and*

$$P(\{u \in \mathbb{R}^{\mathbb{N}}; \lim_{n \to \infty} \eta_u(u) < t\}) = m\,(y((-\infty, t)))$$

for each $t \in \mathbb{R}$.

Now we consider the sequence of observables $\tau \circ x, \tau^2 \circ x, \tau^3 \circ x, \ldots$ and ***the Cesaro means*** defined by

$$z_n = \frac{1}{n} \sum_{i=1}^{n} \tau^i \circ x = \overline{h}_n \circ g_n^{-1}$$

where $\overline{h}_n$ is the joint observable of the observables $\tau \circ x, \tau^2 \circ x, \ldots, \tau^n \circ x$.

Proposition 1. Riečan and Jurečková (2005) *Put* $k_{n+1}(u_1, u_2, \ldots, u_{n+1}) = \frac{1}{n} \sum_{i=2}^{n+1} u_i = g_n(u_2, \ldots, u_{n+1})$ *Then* $z_n = h_{n+1} \circ k_{n+1}^{-1}$.

Theorem 3.2. Riečan and Jurečková (2005) *Let* $\xi_n = \frac{1}{n} \sum_{i=1}^{n} \xi \circ T^i = \eta_n \circ T$. *Then the sequence of observables* $(z_n)_n$ *converges m-almost everywhere to an observable z and*

$$P(\{u \in \mathbb{R}^{\mathbb{N}}; \lim_{n \to \infty} \xi_n(u) < t\}) = m\,(z((-\infty, t)))$$

for each $t \in \mathbb{R}$ *Moreover* $z = \tau \circ y$.

The following two theorems were proved for P-observables y, z, i.e. $y(C \cap D) \leq y(C) \cdot y(D)$ and $z(C \cup D) \leq z(C) \cdot z(D)$ for each $C, D \in \mathcal{B}(\mathbb{R})$ in paper

(Riečan and Jurečková, 2005). We show that the assumption of P-observable can be omitted.

Theorem 3.3. *Let y, z be observables in $\mathcal{F}$ such that $z = \tau \circ y$, $\tau : \mathcal{F} \to \mathcal{F}$ be the σ-homomorphism with properties [T1]–[T6] and m be a state on $\mathcal{F}$. Then for all $t \in \mathbb{R}$ it holds:*

$$m\left(y((-\infty, t)) \cdot z((t, \infty))\right) = 0 \text{ and } m\left(y((t, \infty)) \cdot z((-\infty, t))\right) = 0$$

Proof: Evidently

$$m\left(y((-\infty, t)) \cdot z((t, \infty))\right) = m\left(y((-\infty, t)) \cdot \bigvee_{n=1}^{\infty} z\left(\left\langle t + \frac{1}{n}, \infty\right)\right)\right).$$

Therefore it is sufficient to prove

$$m\left(y((-\infty, t)) \cdot z(\langle s, \infty))\right) = 0$$

for $t < s$.

Of course,

$$m\left(y((-\infty, t)) \cdot z(\langle s, \infty))\right) \leq m\left(y((-\infty, t)) \wedge z(\langle s, \infty))\right)$$

and

$$\begin{aligned} m\left(y((-\infty, t)) \wedge z(\langle s, \infty))\right) &= m\left(y((-\infty, t)) \wedge (1_{\mathcal{F}} - z((-\infty, s)))\right) = \\ &= m\left(y((-\infty, t))\right) - m\left(y((-\infty, t)) \wedge z(\langle s, \infty))\right) \end{aligned}$$

By *Theorem 3.1* we have that

(1) $m\left(y((-\infty, t))\right) = P(\{u \in \mathbb{R}^{\mathbb{N}}; \lim_{n\to\infty} \eta_n(u) < t\}) = P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\})$ where η is the random variable from individual ergodic theorem.

Now we prove that

(2) $m\left(y((-\infty, t)) \wedge z((-\infty, s))\right) \geq P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\} \cap \{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) < s\})$

where $\xi = \lim_{n\to\infty} \xi_n(u)$ is the random variable from individual ergodic theorem.

We know that

$$y((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{i=1}^{\infty} \bigwedge_{n=k}^{k+i} y_n\left(\left(-\infty, t - \frac{1}{p}\right)\right)$$

$$z((-\infty, s)) = \bigvee_{q=1}^{\infty} \bigvee_{l=1}^{\infty} \bigwedge_{j=1}^{\infty} \bigwedge_{m=l}^{l+j} z_m\left(\left(-\infty, s - \frac{1}{q}\right)\right)$$

Therefore

$$\begin{aligned} &m(y((-\infty, t)) \wedge z((-\infty, s))) \\ &\quad = \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \lim_{q\to\infty} \lim_{l\to\infty} \lim_{j\to\infty} m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\infty, t-\frac{1}{p}\right)\right)\right. \\ &\qquad \left.\wedge \bigwedge_{m=l}^{l+j} z_m\left(\left(-\infty, s-\frac{1}{q}\right)\right)\right). \end{aligned}$$

Moreover

$$\begin{aligned} &m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(-\infty, t-\frac{1}{p}\right)\right) \wedge \bigwedge_{m=l}^{k+i} z_m\left(\left(-\infty, s-\frac{1}{q}\right)\right)\right) \\ &\quad = m\left(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}\left(\left(-\infty, t-\frac{1}{p}\right)\right) \wedge \bigwedge_{m=l}^{l+j} h_{m+1} \circ k_{m+1}^{-1}\left(\left(-\infty, s-\frac{1}{q}\right)\right)\right) \\ &\quad = m\left(\bigwedge_{n=k}^{k+i} h_w(A_n) \wedge \bigwedge_{m=l}^{l+k} h_w(B_m)\right) \end{aligned}$$

where $w \geq k+i$, $w \geq l+j$, $A_n = \pi_{w,n}^{-1}(g_n^{-1}((-\infty, t-\frac{1}{p})))$, $B_m = \pi_{w,m+1}^{-1}(g_n^{-1}((-\infty, s-\frac{1}{q})))$ and t, s, p, q are constants.

By monotonicity of h_w we obtain

$$h_w(A_n) \geq h_w\left(\bigcap_{n=k}^{k+i} A_n\right), n = k, \ldots, k+i$$

hence

$$\bigwedge_{n=k}^{k+i} h_w(A_n) \geq h_w\left(\bigcap_{n=k}^{k+i} A_n\right)$$

and similarly

$$\bigwedge_{m=l}^{l+j} h_w(B_m) \geq h_w\left(\bigcap_{m=l}^{l+j} B_m\right).$$

By these relations we obtain

$$\bigwedge_{n=k}^{k+i} h_w(A_n) \wedge \bigwedge_{m=l}^{l+j} h_w(B_m) \geq h_w\left(\bigcap_{n=k}^{k+i} A_n\right) \wedge h_w\left(\bigcap_{m=l}^{l+j} B_m\right)$$

$$\geq h_w\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right).$$

Therefore

$$m\left(\bigwedge_{n=k}^{k+i} y_n\left(\left(\infty, t-\frac{1}{p}\right)\right) \wedge \bigwedge_{m=l}^{l+j} z_m\left(\left(-\infty, s-\frac{1}{q}\right)\right)\right) \geq$$

$$\geq m\left(h_w\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right)\right) = P\left(\Pi_w^{-1}\left(\left(\bigcap_{n=k}^{k+i} A_n\right) \cap \left(\bigcap_{m=l}^{l+j} B_m\right)\right)\right)$$

$$= P\left(\bigcap_{n=k}^{k+i}\left\{u \in \mathbb{R}^{\mathbb{N}}; g_n(u_1, \ldots, u_n) < t - \frac{1}{p}\right\}\right.$$

$$\left.\cap \bigcap_{m=l}^{l+j}\left\{u \in \mathbb{R}^{\mathbb{N}}; k_{m+1}(u_1, \ldots, u_{m+1}) < s - \frac{1}{q}\right\}\right)$$

hence

$$m\left(y((-\infty, t)) \wedge z((-\infty, s))\right) \geq \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \lim_{q\to\infty} \lim_{l\to\infty} \lim_{j\to\infty}$$

$$P\left(\bigcap_{n=k}^{k+i}\left\{u \in \mathbb{R}^{\mathbb{N}}; g_n(u_1, \ldots, u_n) < t - \frac{1}{p}\right\}\right.$$

$$\left.\cap \bigcup_{m=1}^{l+j}\left\{u \in \mathbb{R}^{\mathbb{N}}; k_{m+1}(u_1, \ldots, u_{m+1}) < s - \frac{1}{q}\right\}\right)$$

$$= P\left(\bigcup_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\left\{u \in \mathbb{R}^{\mathbb{N}}; \eta_n(u) < t - \frac{1}{p}\right\}\right.$$

$$\left.\bigcap \bigcup_{q=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\left\{u \in \mathbb{R}^{\mathbb{N}}; \xi_n(u) < s - \frac{1}{q}\right\}\right)$$

$$= P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\} \cap \{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) < s\}).$$

By (1) and (2) we obtain

$$
\begin{aligned}
m\,(y((-\infty, t)) \cdot z(\langle s, \infty))) &\leq m\,(y((-\infty, t)) \wedge z(\langle s, \infty))) = \\
&= m\,(y((-\infty, t))) - m\,(y((-\infty, t)) \wedge z((-\infty, s))) \leq \\
&\leq P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\}) - \\
&- P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\} \cap \{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) < s\}) \\
&= P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\} \cap \{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) \geq s\})
\end{aligned}
$$

Since $\eta = \xi = \eta \circ T\, P$ almost everywhere by individual ergodic theorem, then

$$P(\{u \in \mathbb{R}^{\mathbb{N}}; \eta(u) < t\} \cap \{u \in \mathbb{R}^{\mathbb{N}}; \xi(u) \geq s\}) = 0.$$

Hence

$$m\,(y((-\infty, t)) \cdot z(\langle s, \infty))) = 0$$

Theorem 3.4. $y = z = \tau \circ y$ *m-almost everywhere.*

Proof: It follows by Theorem 2.1 and Theorem 3.3. □

4. CONCLUSION

The paper is concerned in ergodic theory for fuzzy quantum logic $\mathcal{F}$. We proved that the limit of Cesaro means is an invariant observable and show that the assumption of P-observable is redundant in this case.

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